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XXXVI. *A Disquisition concerning certain Fluents, which are assignable by the Arcs of the Conic Sections ; wherein are investigated some new and useful Theorems for computing such Fluents : By John Landen, F. R. S.*

Read June 6,
1771.

MR. Mac Laurin, in his Treatise of Fluxions, has given sundry very elegant Theorems for computing the Fluents of certain Fluxions by means of Elliptic and Hyperbolic Arcs ; and Mr. D'Alembert, in the Memoirs of the Berlin Academy, has made some improvement upon what had been before written on that subject, But some of the Theorems given by those Gentlemen being in part expressed by the difference between an Arc of an Hyperbola and its Tangent, and such difference being not directly attainable, when such Arc and its Tangent both become infinite, as they will do when the *whole* Fluent is wanted, although such Fluent be at the same time finite ; those Theorems therefore in that case fail, a computation thereby being then impracticable, without some farther help.

The supplying that defect I considered as a point of some importance in Geometry, and therefore I

earnestly wished, and endeavoured, to accomplish that business; my aim being to ascertain, by means of such arcs, as above-mentioned, the *Limit* of the difference between the Hyperbolic Arc and its Tangent, whilst the point of contact is supposed to be carried to an infinite distance from the vertex of the curve, seeing that, by the help of that *Limit*, the computation would be rendered practicable in the case wherein, without such help, the before-mentioned Theorems fail. And having succeeded to my satisfaction, I presume, the result of my endeavours, which this Paper contains, will not be unacceptable to the Royal Society.

I.

Suppose the curve ADEF (Tab. XII. fig. 1.) to be a conic *Hyperbola*, whose semi-transverse axis AC is $= m$, and semi-conjugate $= n$.

Let CP, perpendicular to the tangent DP, be called p ; and put $f = \frac{m^2 - n^2}{2m}$, $x = \frac{p^2}{m}$. Then (as is well known) will DP — AD be $=$ the fluent of $\frac{-\frac{1}{2}m^{\frac{1}{2}}x^{\frac{1}{2}}}{\sqrt{n^2 + 2fx - x^2}}$, p and x being each $=$ to m when AD is $= c$.

2.

Suppose the curve adefg (fig. 2.) to be a quadrant of an *Ellipsis*, whose semi-transverse axis cg is $= \sqrt{m^2 + n^2}$, and semi-conjugate ac $= n$. Let

Q q 2 ct

ct be perpendicular to the tangent dt, and let the abscissa cp be $= n \times \frac{z}{m}^{\frac{1}{2}}$. Then will the said tangent dt be $= m \times \frac{mz - z^2}{n^2 + mz}^{\frac{1}{2}}$; and the fluxion thereof will be found

$$= \frac{1}{2} mn^2 z^{-\frac{1}{2}} \dot{z} \times \frac{\overline{m - z}^{\frac{1}{2}}}{\overline{n^2 + mz}^{\frac{3}{2}}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$$

3.

In the expreffion $\frac{y q^{-1} \dot{y}}{a + by|^r \times c + dy|^s}$, let $\frac{c + dy}{a + by}$ be supposed $= z$. Then will $\frac{az - c}{d - bz}$ be $= y$, and the proposed expreffion will be

$$= \frac{\overline{aa - bd}^{1-r-s} \times z^{-s} \dot{z}}{\overline{az - c}^{1-q} \times \overline{d - bz}^{1+q-r-s}}$$

4.

Taking, in the last article, r and s each $= \frac{1}{2}$, $q = \frac{3}{2}$, $a = -d = \frac{n^2}{m}$, $b = 1$, and $c = n^2$, we have

$$\frac{y^{\frac{1}{2}} \dot{y}}{\left(\frac{n^2}{m} + y \right)^{\frac{1}{2}} \times \overline{n^2 - \frac{n^2}{m} y}^{\frac{1}{2}}} \left(= \frac{m^{\frac{1}{2}} n^{-1} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} \right)$$

$$= -mnz^{-\frac{1}{2}} \dot{z} \times \frac{\overline{m - z}^{\frac{1}{2}}}{\overline{n^2 + mz}^{\frac{3}{2}}}$$

It

It appears therefore, that, y being $= n^2 \times \frac{m-z}{n^2+mz}$,

$$-\frac{\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} \text{ is}$$

$$= \frac{1}{2} m n^2 z^{-\frac{1}{2}} \dot{z} \times \frac{(m-z)^{\frac{1}{2}}}{(n^2+mz)^{\frac{3}{2}}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}};$$

which, by Art. 2. is = the *fluxion of the tang.* dt.

Consequently, taking the fluents, by Art. 1. and correcting them properly, we find

$$DP - AD + FR - AF = L + dt.$$

$$CP, \text{ in fig. 1. being } = m^{\frac{1}{2}} z^{\frac{1}{2}}; \text{ cp, in fig. 2. } = n \times \left[\frac{z}{m} \right]^{\frac{1}{2}};$$

$$CR, \text{ perpendicular to the tangent } FR = m^{\frac{1}{2}} y^{\frac{1}{2}};$$

$$DP - AD = \text{the fluent of } \frac{-\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}};$$

$$FR - AF = \text{the fluent of } \frac{-\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}};$$

and L the *Limit* to which the difference $DP - AD$, or $FR - AF$, approaches, upon carrying the point D , or F , from the vertex A *ad infinitum*.

5.

Suppose y equal to z , and that the points D and F then coincide in E , the points d and p being at the same time in e and q respectively. Then cv being perpendicular to the tangent ev , that tangent will be a *maximum* and equal to $cg - ac = \sqrt{m^2 + n^2} - n$; the tangent EQ (in the hyperbola) will be $= \sqrt{m^2 + n^2}$;
the

the abscissa $BC = m \sqrt{1 + \frac{n^2}{\sqrt{m^2 + n^2}}}$; the ordinate

$BE = n \times \sqrt{\frac{n}{\sqrt{m^2 + n^2}}}$; and it appears, that

L is $= 2EQ - 2AE - cv = n + \sqrt{m^2 + n^2} - 2AE$!
Thus the *Limit* which I proposed to ascertain is investigated, m and n being any right lines whatever!

6.

The whole fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes $= m$, being equal to L ; and the fluent of the same fluxion (supposing it to begin when z begins) being in general equal to $L + AD - DP = FR - AF - dt$; it appears, that, k being the value of z corresponding to the fluent $L + AD - DP$, $\frac{mn^2 - n^2 k}{n^2 + mk}$ will be the value of z corresponding to the fluent $L + AF - FR$, and $FR - AF$ will be the part generated whilst z from $\frac{mn^2 - n^2 k}{n^2 + mk}$ becomes $= m$. It follows therefore, that the *tang.* dt , together with the fluent of $\frac{\frac{3}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}}$ generated whilst z from o becomes equal to any quantity k , is equal to the fluent of the same fluxion generated whilst z from $\frac{mn^2 - n^2 k}{n^2 + mk}$ becomes $= m$; cp being taken $= n \times \frac{k^{\frac{1}{2}}}{m}$.

Suppose

Suppose $k = \frac{m n^2 - n^2 k}{n^2 + m k}$, its value will then be

$\frac{n}{m} \sqrt{m^2 + n^2} - \frac{n^2}{m}$. Consequently the fluent of

$\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}}$ generated whilst z from o becomes

$= \frac{n}{m} \sqrt{m^2 + n^2} - \frac{n^2}{m}$, together with the quantity

$\sqrt{m^2 + n^2} - n$, is equal to the fluent of the same

fluxion generated whilst z from $\frac{n}{m} \sqrt{m^2 + n^2} - \frac{n^2}{m}$

becomes $= m$: and these two parts of the *whole fluent* being denoted by M and N respectively;

M will be $= n - A E$, and $N = \sqrt{m^2 + n^2} - A E$.

7.

The fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}}$ being $L + A D - D P$,

the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}} + D P - A D - L$ will be $= a$.

Therefore, the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}} +$ the fluent of

$\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}}$ being $=$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \frac{n^2 + m z}{m - z}^{\frac{1}{2}}$,

it is obvious, that the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2 f z - z^2}}$ is

$= D P - A D - L +$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \frac{n^2 + m z}{m - z}^{\frac{1}{2}}$
 $= D P$

$= DP - AD - L + \text{the elliptic arc } dg \text{ (fig. 2.)}$

whose abscissa cp is $= n \times \frac{z}{m}^{\frac{1}{2}}$.

Consequently, putting E for $\frac{1}{4}$ of the periphery of that ellipsis, it appears that the *whole fluent* of

$\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes $= m$, is equal to $E - L = E + 2AE - n - \sqrt{m^2 + n^2}$.

8.

By taking, in Art. 3. q, r , and s , each $= \frac{1}{2}$; and $a = -d = \frac{n^2}{m}$, $b = 1$, and $c = n^2$; we find,

that, if y be $= \frac{mn^2 - n^2 z}{n^2 + mz}$, $\frac{z^{-\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}} + \frac{y^{-\frac{1}{2}} y}{\sqrt{n^2 + 2fy - y^2}}$ will be $= o$.

It is obvious therefore, that the fluent of $\frac{z^{-\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes equal to any quantity k , is equal to the fluent of the same fluxion, generated whilst z from $\frac{mn^2 - n^2 k}{n^2 + mk}$ becomes $= m$.

Now, supposing $k = \frac{mn^2 - n^2 k}{n^2 + mk}$, its value will be $\frac{n}{m} \sqrt{m^2 + n^2} - \frac{n^2}{m}$.

Consequently, the fluent of $\frac{z^{-\frac{1}{2}} z}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes $= \frac{n}{m} \sqrt{m^2 + n^2} - \frac{n^2}{m}$,
is

is equal to *half* the fluent of the same fluxion, generated whilst z from o becomes $= m$; which *half fluent* is known by the preceding article.

9.

It appears, by Ar. 4. that

$$\frac{\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} \text{ is } = - \text{the flux. of the tang. dt};$$

and it appears, by the last article, that

$$\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 y^{-\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} \text{ is } = 0;$$

$m n^2 - n^2 y - n^2 z - m y z$ being $= 0$.

Therefore, by addition, we have

$$\frac{1}{2} y^{-\frac{1}{2}} \dot{y} \times \left[\frac{n^2 + m y}{m - y} \right]^{\frac{1}{2}} + \frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \left[\frac{n^2 + m z}{m - z} \right]^{\frac{1}{2}} \\ = - \text{the fluxion of the tangent dt.}$$

Consequently, by taking the correct fluents, we find the *tang. dt* ($=$ the *tang. fw*) $=$ the *arc* ad — the *arc fg*! the abscissa cp being $= n \times \left[\frac{z}{m} \right]^{\frac{1}{2}}$,

the abscissa $cr = n \times \left[\frac{y}{m} \right]^{\frac{1}{2}}$, and their relation expressed by the equation $n^6 - n^4 u^2 - n^4 v^2 - m^2 u^2 v^2 = 0$, u and v being put for cp and cr respectively.

Moreover, the tangents dt , fw , will each be $= \frac{m^2 uv}{n^4}$;

and $ct \times cw = \overline{cv^2} = ac \times cg$.

If for the semi-transverse axis cg we substitute b instead of $\sqrt{m^2 + n^2}$, the relation of u to v will be

expressed by the equation

$$n^6 - n^4 u^2 - n^4 v^2 - \overline{b^2 - n^2} \times u^2 v^2 = 0, \text{ and}$$

$$dt (=fw) \text{ will be } = \frac{b^2 - n^2}{n^3} \times uv.$$

If u and v be respectively put for fr and dp , their relation will be expressed by the equation

$$b^6 - b^4 u^2 - b^4 v^2 + \overline{b^2 - n^2} \times u^2 v^2 = 0, \text{ and}$$

$$dt (=fw) \text{ will be } = \frac{b^2 - n^2}{b^3} \times uv.$$

10.

Suppose $y =$ to z , (that is, $v = u$) and that the points d and f coincide in e . In which case the tangent dt will be a *maximum*, and $= cg - ac$. It appears then that the *arc* ae — the *arc* eg is $= cg - ac$.

Consequently, putting E for the quadrantal arc ag , we find that the *arc* ae is $= \frac{E + b - n}{2}$!

$$\text{the arc } eg = \frac{E - b + n}{2}!$$

There are, I am aware, some other parts of the *arc* ag , whose lengths may be assigned by means of the whole length (ag) with right lines; but to investigate such other parts is not to my present purpose.

11.

Taking m and n each $= 1$; that is, $ac (=AC) = 1$, and $cg = \sqrt{2}$; let the *arc* ag be then expressed

pressed by e : put c for $\frac{1}{4}$ of the periphery of the circle whose radius is 1; and let the *whole fluents* of

$\frac{\frac{1}{2} z^{\frac{1}{2}} \dot{z}}{\sqrt{1-z^2}}$ and $\frac{\frac{1}{2} z^{-\frac{1}{2}} \dot{z}}{\sqrt{1-z^2}}$, generated whilst z from 0 be-

comes $= 1$, be denoted by F and G respectively.

Then, by what is said above, $F + G$ will be $= e$; and, by my theorem for comparing curvilineal areas, or fluents, published in the *Philos. Transact.* for the year 1768, it appears that $F \times G$ is $= \frac{1}{2}c$. From

which equations we find $F = \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$, and $G = \frac{1}{2}e + \frac{1}{2}\sqrt{e^2 - 2c}$.

But m and n being each $= 1$, L is $= F$; therefore $1 + \sqrt{2} - 2AE$, the value of L , from Art. 5. is, in this case, $= \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$. Consequently, in the *equilateral hyperbola*, the *arc* AE , whose abscissa

BC is $= \sqrt{1 + \frac{1}{\sqrt{2}}}$, will be $= \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{4}e + \frac{1}{4}\sqrt{e^2 - 2c}$,

by what is said in the article last mentioned. Hence the *rectification* of that *arc* may be effected by means of the *circle* and *ellipsis*!

The application of these *Improvements* will be easily made by the intelligent Reader, who is acquainted with what has been before written on the subject. But there is a theorem (demonstrable by what is proved in Art. 8.) so remarkable, that I cannot conclude this disquisition without taking notice of it.

Let $lpqn$ (fig. 3.) be a circle perpendicular to the horizon, whose highest point is l , lowest n , and center m : let p and q be any points in the semicircumference $lpqn$: draw ps , qt parallel to the horizon, intersecting lmn in s and t ; and, having joined lp , pt , make the angle lpv equal to ltp , and draw rv parallel to qt , intersecting the circle in r , and the diameter lmn in v . Let a pendulum, or other heavy body, descend by its gravity from p along the arc $pqrn$: the body so descending will pass over the arc pq exactly in the same time as it will pass over the arc rn ; and therefore, qt and rv coinciding when lt is equal to lp , it is evident that the time of descent from p to q will then be precisely equal to *half* the time of descent from p to n !

And it is farther observable, that, if pqn be a quadrant, the *whole* time of descent will be

$$= \left(\frac{a}{b} \right)^{\frac{1}{2}} \times \frac{\frac{1}{2}e + \frac{1}{2}\sqrt{e^2 - 2c}}{1}; \text{ the radius } lm, \text{ or } mn,$$

being $=a$; and b being put (for $16\frac{1}{2}$ feet) the space a heavy body descending freely from rest falls through in one second of time.

In general, ns being denoted by d , and the distance of the body from the line ps , in its descent, by x , the fluxion of the time of descent will be expressed by

$$\frac{\frac{1}{2}ab^{-\frac{1}{2}}x^{-\frac{1}{2}}}{\sqrt{2ad - d^2 - 2a - 2d \cdot x - x^2}}; \text{ the fluent whereof,}$$

corresponding to any value of x , may be obtained by Art. 7. By which article it appears, that the *whole* time

time of descent from any point p will be

$$= \frac{a}{b^{\frac{1}{2}} d^{\frac{1}{2}} \times 2a - d} \times \overline{E + 2AE - pn - ps}.$$

The semi-transverse AC (fig. 1.) being $= ns$;
 the semi-transverse cg (fig. 2.) $= np$;
 and the semi-conjugate in each figure $= ps$.

Since writing the above, I have discovered a general theorem for the rectification of the Hyperbola, by means of two Ellipses; the investigation whereof I purpose to make the subject of another Paper.